Available online at www.sciencedirect.com
SCIENCE @ DIRECT®
Journal of
MATHEMATICAL

www.elsevier.com/locate/jmaa

Asymptotic behavior of solutions of functional difference equations

Hideaki Matsunaga [a,1], Satoru Murakami [b,2,*]

[a] Department of Mathematical Sciences, Osaka Prefecture University, Sakai 599-8531, Japan
[b] Department of Applied Mathematics, Okayama University of Science, Okayama 700-0005, Japan

Received 8 July 2004

Available online 8 January 2005

Submitted by T. Krisztin

Abstract

For linear functional difference equations, we obtain some results on the asymptotic behavior of solutions, which correspond to a Perron-type theorem for linear ordinary difference equations. We also apply our results to Volterra difference equations with infinite delay.

Keywords: Functional difference equations; Asymptotic behavior; Phase space; Volterra difference equations; Delay

1. Introduction

Let $\mathbf{Z}$, $\mathbf{Z}^+$ and $\mathbf{Z}^-$ be the set of all integers, the set of all nonnegative integers and the set of all nonpositive integers, respectively. Denote by $\mathbf{C}^k$ the k-dimensional complex

* Corresponding author.
E-mail addresses: hideaki@ms.osakafu-u.ac.jp (H. Matsunaga), murakami@xmath.ous.ac.jp (S. Murakami).
[1] Supported in part by Grant-in-Aid for Young Scientists (B), No. 15740109, of the Japanese Ministry of Education, Culture, Sports, Science and Technology.
[2] Supported in part by Grant-in-Aid for Scientific Research (C), No. 16540177, of the Japanese Ministry of Education, Culture, Sports, Science and Technology.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2004.10.065

Euclidean space with any convenient norm $|\cdot|$. For any function $x:(-\infty,m]\to \mathbf{C}^k$ and any $n\in\mathbf{Z}$ with $n\leqslant m$, we define a function $x_n:\mathbf{Z}^-\to\mathbf{C}^k$ by $x_n(s)=x(n+s)$ for $s\in\mathbf{Z}^-$.

In this paper we are concerned with the linear functional difference equation

$$x(n+1)=L(x_n) \tag{1.1}$$

and its perturbed equation

$$x(n+1)=L(x_n)+G(n)x_n, \tag{1.2}$$

where L and $G(n)$, $n\in\mathbf{Z}^+$, are bounded linear operators from $\mathcal{B}^{\gamma}_{J_l}$ into $\mathbf{C}^k$; here and hereafter, $\mathcal{B}^{\gamma}_{J_l}$ is a seminormed linear space equipped with seminorm $\|\cdot\|_{\mathcal{B}^{\gamma}_{J_l}}$ (which will be introduced in Section 2) and $0\leqslant l\leqslant\infty$. The theory of functional difference equations, together with Volterra difference equations, has recently been studied by several authors; see [1–6,8,9] and references therein.

The purpose of this paper is to investigate the asymptotic behavior of solutions of Eq. (1.2) under the condition

$$\lim_{n\to\infty}\bigl\|G(n)\bigr\|=0, \tag{1.3}$$

where $\|G(n)\|$ is the operator norm of $G(n)$. In Theorem 2.1, we will establish a result on the limit $\lim_{n\to\infty}\sqrt[n]{\|x_n\|_{\mathcal{B}^{\gamma}_{J_l}}}$ for any solution x of Eq. (1.2). Our result is intimately related to a result [10, Theorem 1] due to Pituk for linear *ordinary* difference equations. In fact, as will be explained in Section 2, linear ordinary difference equations treated in [10] are viewed as special ones of Eq. (1.2) on $\mathcal{B}^{\gamma}_{J_l}$ with $l=0$, and [10, Theorem 1] can be derived from Theorem 2.1 with $l=0$. Also, for solutions x of Eq. (1.2) we will obtain a result on the limit of $\sqrt[n]{|x(n)|}$ but not $\sqrt[n]{\|x(n)\|_{\mathcal{B}^{\gamma}_{J_l}}}$ (Theorem 2.2). Finally, we will treat linear Volterra difference equations with infinite delay or unbounded delay, and applying Theorem 2.2 we will derive some results on the asymptotic behavior of solutions for the equations, assuming some mild conditions on the kernels.

2. Some notations and statements of main results

For any $l\in\mathbf{Z}^+$ or $l=\infty$, we denote by J_l the discrete interval $\{-l,-l+1,\dots,0\}=:[-l,0]$ in $\mathbf{Z}^-$. In the above, it is understood that $J_\infty=\mathbf{Z}^-$ in case of $l=\infty$. For a fixed interval J_l and a constant γ such that $\gamma\geqslant 1$, we consider the linear space $\mathcal{B}^{\gamma}_{J_l}$ defined by

$$\mathcal{B}^{\gamma}_{J_l}:=\Bigl\{\phi:\mathbf{Z}^-\to\mathbf{C}^k \;\Big|\; \sup_{s\in J_l}\bigl|\phi(s)\bigr|\gamma^s<\infty\Bigr\}$$

equipped with the seminorm norm

$$\|\phi\|_{\mathcal{B}^{\gamma}_{J_l}}=\sup_{s\in J_l}\bigl|\phi(s)\bigr|\gamma^s.$$

Clearly, the seminormed linear space $\mathcal{B}^{\gamma}_{J_l}$ is complete; that is, the quotient $\mathcal{B}^{\gamma}_{J_l}/\|\cdot\|_{\mathcal{B}^{\gamma}_{J_l}}=:\hat{\mathcal{B}}^{\gamma}_{J_l}$ is a Banach space. In particular, $\mathcal{B}^{\gamma}:=\mathcal{B}^{\gamma}_{J_\infty}$ itself is a Banach space. Recall that for any function $x:\mathbf{Z}\to\mathbf{C}^k$ and any $n\in\mathbf{Z}$, the segment $x_n:\mathbf{Z}^-\to\mathbf{C}^k$ is a function defined by $x_n(s)=x(n+s)$ for $s\in\mathbf{Z}^-$. It is easy to see that if $x_0\in\mathcal{B}^\gamma_{J_l}$, then $x_n\in\mathcal{B}^\gamma_{J_l}$ for any $n\in\mathbf{Z}^+$.

A bounded linear operator $L:\mathcal{B}^\gamma_{J_l}\to\mathbf{C}^k$ is said to be bounded, if $\|L\|:=\sup\{|L(\phi)|: \|\phi\|_{\mathcal{B}^\gamma_{J_l}}\leqslant 1\}$ is finite. In order to state our main results on the asymptotic behavior of solutions of Eq. (1.2) with bounded linear operators L and $G(n):\mathcal{B}^\gamma_{J_l}\to\mathbf{C}^k$, we need the characteristic matrix and the characteristic equation of Eq. (1.1) defined by

$$\Delta(z):=zI-L(\omega_z I),\quad |z|>\frac{1}{\gamma},$$

$$\det\Delta(z)=\det\bigl(zI-L(\omega_z I)\bigr)=0,\quad |z|>\frac{1}{\gamma},$$

respectively, where I denotes the $k\times k$ identity matrix and ω_z is defined as $\omega_z(s)=z^s$ for $s\in\mathbf{Z}^-$. The following theorems are our main results.

Theorem 2.1. *Suppose* (1.3) *holds. If x is a solution of Eq.* (1.2)*, then either*

$$\limsup_{n\to\infty}\sqrt[n]{\|x_n\|_{\mathcal{B}^\gamma_{J_l}}}\leqslant\frac{1}{\gamma}$$

or

$$\lim_{n\to\infty}\sqrt[n]{\|x_n\|_{\mathcal{B}^\gamma_{J_l}}}=|\lambda|,$$

where λ is a root of $\det\Delta(\lambda)=0$ *with* $|\lambda|>1/\gamma$.

Theorem 2.2. *Suppose* (1.3) *holds. If x is a solution of Eq.* (1.2)*, then either*

$$\limsup_{n\to\infty}\sqrt[n]{\bigl|x(n)\bigr|}\leqslant\frac{1}{\gamma}$$

or

$$\limsup_{n\to\infty}\sqrt[n]{\bigl|x(n)\bigr|}=|\lambda|,\tag{2.1}$$

where λ is a root of $\det\Delta(\lambda)=0$ *with* $|\lambda|>1/\gamma$.

In the following, we will state some remarks concerning Theorems 2.1 and 2.2. Let us consider the ordinary difference equation

$$x(n+1)=\bigl[A+B(n)\bigr]x(n),\tag{2.2}$$

where A and $B(n)$ are $k\times k$ complex matrices and $\lim_{n\to\infty}\|B(n)\|=0$. In [10, Theorem 1], Pituk has proved that if x is a solution of Eq. (2.2), then $x(n)=0$ for all large n or $\lim_{n\to\infty}\sqrt[n]{|x(n)|}=|\lambda|$, where λ is an eigenvalue of the matrix A. In fact, Theorem 2.1 generalizes [10, Theorem 1]. Indeed, putting $L(\phi)=A\phi(0)$ and $G(n)\phi=B(n)\phi(0)$ for $\phi\in\mathcal{B}^\gamma_{J_0}$, one can view Eq. (2.2) as Eq. (1.2) on $\mathcal{B}^\gamma_{J_0}$ with any $\gamma\geqslant 1$. Notice that $\|\phi\|_{\mathcal{B}^\gamma_{J_0}}=|\phi(0)|$. Then $\limsup_{n\to\infty}\sqrt[n]{\|x_n\|_{\mathcal{B}^\gamma_{J_0}}}\leqslant 1/\gamma$ for any $\gamma\geqslant 1$ means that $\lim_{n\to\infty}\sqrt[n]{|x(n)|}=0$. By combining this fact with the following lemma, one can see that [10, Theorem 1] follows from Theorem 2.1.

Lemma 2.1. *Suppose that* $\lim_{n\to\infty}\|B(n)\|=0$*, and that Eq.* (2.2) *has a nontrivial solution* x *such that* $\lim_{n\to\infty}\sqrt[n]{|x(n)|}=0$*. Then the matrix* A *is singular.*

Proof. Assume that A is nonsingular, and choose an $n_0\in\mathbf{Z}^+$ so that $|x(n_0)|\neq 0$ and that $\|B(n)\|<1/(2\|A^{-1}\|)$ for $n\geqslant n_0$. Let $n\geqslant n_0$. One can easily check that $(I+A^{-1}B(n))$ is nonsingular with $\|(I+A^{-1}B(n))^{-1}\|<2$, and hence $|x(n)|=|(I+A^{-1}B(n))^{-1}A^{-1}x(n+1)|\leqslant 2\|A^{-1}\||x(n+1)|$. It follows that $|x(n)|\neq 0$ and

$$\left|x(n)\right|=\frac{|x(n)|}{|x(n-1)|}\cdots\frac{|x(n_0+1)|}{|x(n_0)|}\cdot\left|x(n_0)\right|\geqslant\left(\frac{1}{2\|A^{-1}\|}\right)^{n-n_0}\left|x(n_0)\right|,$$

which shows that $\liminf_{n\to\infty}\sqrt[n]{|x(n)|}\geqslant 1/(2\|A^{-1}\|)>0$. This is a contradiction to $\lim_{n\to\infty}\sqrt[n]{|x(n)|}=0$. Thus, A must be singular. □

We also remark that the "lim sup" in the relation (2.1) of Theorem 2.2 cannot always be replaced with the limit. To see this, let us consider the scalar equation

$$x(n+1)=x(n-1),$$

which is a modification of the equation given in [10, p. 205]. Putting $L(\phi)=\phi(-1)$ for $\phi\in\mathcal{B}^{\gamma}_{J_l}$ with $\gamma>1$ and $l\geqslant 1$, one can consider the above equation as a functional difference equation (1.1). Notice that the roots of the characteristic equation are $\lambda_1=1$ and $\lambda_2=-1$. Clearly, the equation has the solution $x(n)=1+(-1)^n$, for which $\limsup_{n\to\infty}\sqrt[n]{|x(n)|}=1=|\lambda_1|=|\lambda_2|$, but $\lim_{n\to\infty}\sqrt[n]{|x(n)|}$ does not exist.

We next consider the Volterra difference equation with infinite delay

$$x(n+1)=\sum_{s=-\infty}^{n}\left\{Q(n-s)+g(n,s)\right\}x(s), \tag{2.3}$$

where $Q(n-s)$ and $g(n,s)$ are $k\times k$ complex matrices defined for $n\in\mathbf{Z}^+$, $s\in\mathbf{Z}$ with $n\geqslant s$, and satisfy

$$\sum_{m=0}^{\infty}\left\|Q(m)\right\|\gamma^m<\infty, \tag{2.4}$$

$$\sum_{m=0}^{\infty}\left\|g(n,n-m)\right\|\gamma^m<\infty\quad\text{and}\quad\lim_{n\to\infty}\left(\sum_{m=0}^{\infty}\left\|g(n,n-m)\right\|\gamma^m\right)=0 \tag{2.5}$$

for some $\gamma\geqslant 1$, respectively. Eq. (2.3) can be viewed as Eq. (1.2) on $\mathcal{B}^{\gamma}:=\mathcal{B}^{\gamma}_{J_\infty}$ with bounded linear operators

$$L(\phi)\equiv\sum_{m=0}^{\infty}Q(m)\phi(-m),\quad \phi\in\mathcal{B}^{\gamma},$$

$$G(n)\phi\equiv\sum_{m=0}^{\infty}g(n,n-m)\phi(-m),\quad \phi\in\mathcal{B}^{\gamma}.$$

Observe that $\|G(n)\|\leqslant\sum_{m=0}^{\infty}\|g(n,n-m)\|\gamma^m$, and consequently condition (1.3) is satisfied by (2.5). Hence we have the following result as a corollary of Theorem 2.2.

Corollary 2.1. *Suppose* (2.4) *and* (2.5) *hold. If x is a solution of Eq.* (2.3)*, then either*

$$\limsup_{n\to\infty} \sqrt[n]{|x(n)|} \leqslant \frac{1}{\gamma}$$

or

$$\limsup_{n\to\infty} \sqrt[n]{|x(n)|} = |\lambda|,$$

where λ is a root of $\det(\lambda I - \sum_{m=0}^{\infty} Q(m)\lambda^{-m}) = 0$ *with* $|\lambda| > 1/\gamma$.

Theorem 2.2 is applicable also to the Volterra difference equation with unbounded delay

$$x(n+1) = \sum_{s=0}^{n} \bigl\{Q(n-s) + g(n,s)\bigr\} x(s). \tag{2.6}$$

Indeed, Eq. (2.6) can be viewed as Eq. (1.2) on $\mathcal{B}^{\gamma}$ with bounded linear operators

$$L(\phi) \equiv \sum_{m=0}^{\infty} Q(m)\phi(-m), \quad \phi \in \mathcal{B}^{\gamma},$$

$$G(n)\phi \equiv -\sum_{s=-\infty}^{-1} Q(n-s)\phi(s-n) + \sum_{s=0}^{n} g(n,s)\phi(s-n), \quad \phi \in \mathcal{B}^{\gamma}.$$

Notice that condition (1.3) is satisfied, because

$$\bigl\|G(n)\bigr\| \leqslant \sum_{m=n+1}^{\infty} \bigl\|Q(m)\bigr\|\gamma^m + \sum_{m=0}^{n} \bigl\|g(n,n-m)\bigr\|\gamma^m \to 0 \quad \text{as } n\to\infty,$$

by (2.4) and (2.5). Thus, we can get the following result.

Corollary 2.2. *Suppose* (2.4) *and* (2.5) *hold. If x is a solution of Eq.* (2.6)*, then either*

$$\limsup_{n\to\infty} \sqrt[n]{|x(n)|} \leqslant \frac{1}{\gamma}$$

or

$$\limsup_{n\to\infty} \sqrt[n]{|x(n)|} = |\lambda|,$$

where λ is a root of $\det(\lambda I - \sum_{m=0}^{\infty} Q(m)\lambda^{-m}) = 0$ *with* $|\lambda| > 1/\gamma$.

3. Some auxiliary results

In this section we summarize some auxiliary results which are essentially used in the proof of our main results.

We consider the nonhomogeneous functional difference equation

$$x(n+1) = L(x_n) + p(n), \tag{3.1}$$

where $L : \mathcal{B}^\gamma_{J_l} \to \mathbf{C}^k$ is a bounded linear operator and $p : \mathbf{Z} \to \mathbf{C}^k$. For any $(\tau, \phi) \in \mathbf{Z} \times \mathcal{B}^\gamma_{J_l}$, there exists a unique function $x : \mathbf{Z} \to \mathbf{C}^k$ such that $x(\tau + s) = \phi(s)$ for any $s \in \mathbf{Z}^-$ and x satisfies Eq. (3.1) for all $n \geqslant \tau$. The function x is called a solution of Eq. (3.1) through (τ, ϕ) and is denoted by $x(\cdot, \tau, \phi; p)$. For any $n \in \mathbf{Z}^+$, we define an operator $T(n)$ on $\mathcal{B}^\gamma_{J_l}$ by

$$\big[T(n)\phi\big](s) = x(n + s, 0, \phi; 0), \quad \phi \in \mathcal{B}^\gamma_{J_l},\ s \in \mathbf{Z}^-.$$

$T(n)$ is called the solution operator of the homogeneous difference equation (1.1). One can easily see that the operator $T(n)$ is bounded and linear, and it satisfies the following semigroup property:

$$T(n)T(m) = T(n + m), \quad n, m \in \mathbf{Z}^+.$$

Therefore, we get the relation

$$T(n) = T^n, \quad n \in \mathbf{Z}^+,$$

where $T = T(1)$.

Let $\Gamma(s)$, $s \in \mathbf{Z}^-$, be a matrix function defined by

$$\Gamma(s) = \begin{cases} I, & s = 0, \\ O, & s = -1, -2, \ldots, \end{cases}$$

where O is the $k \times k$ zero matrix. It can easily be verified that if $y \in \mathbf{C}^k$, then $\Gamma y \in \mathcal{B}^\gamma_{J_l}$ and $\|\Gamma y\|_{\mathcal{B}^\gamma_{J_l}} = |y|$.

The following proposition yields a representation formula for solutions of Eq. (3.1) in the phase space $\mathcal{B}^\gamma_{J_l}$. Here and hereafter, we use the usual convention

$$\sum_\tau^m = 0 \quad \text{for } m < \tau.$$

Proposition 3.1. *Let $(\tau, \phi) \in \mathbf{Z} \times \mathcal{B}^\gamma_{J_l}$ be given. Then the segment $x_n(\tau, \phi; p)$ of solution $x(\cdot, \tau, \phi; p)$ of Eq.* (3.1) *satisfies the following relation in $\mathcal{B}^\gamma_{J_l}$:*

$$x_n(\tau, \phi; p) = T(n - \tau)\phi + \sum_{s=\tau}^{n-1} T(n - s - 1)\big(\Gamma p(s)\big), \quad n \geqslant \tau.$$

In fact, the above representation formula has been established in [8, Theorem 2.1] for the case that the phase space is a Banach space $\mathcal{B}^\gamma$. We emphasize that the method employed in the proof of [8, Theorem 2.1] works well even for the case that the phase space is a seminormed linear space $\mathcal{B}^\gamma_{J_l}$.

Now, let us assume that $\|\phi - \psi\|_{\mathcal{B}^\gamma_{J_l}} = 0$. Then $x(n, \tau, \phi; p) = x(n, \tau, \psi; p)$ and

$$\big\|x_n(\tau, \phi; p) - x_n(\tau, \psi; p)\big\|_{\mathcal{B}^\gamma_{J_l}} = 0 \quad \text{for any } n \geqslant \tau.$$

In particular, we get

$$\big\|T(n)\phi - T(n)\psi\big\|_{\mathcal{B}^\gamma_{J_l}} = 0.$$

Hence there exists uniquely an operator $\hat{T}(n)$ on the quotient space $\hat{\mathcal{B}}^{\gamma}_{J_l}$ which satisfies the relation

$$\hat{T}(n)\Pi = \Pi T(n),$$

where $\Pi : \mathcal{B}^{\gamma}_{J_l} \to \hat{\mathcal{B}}^{\gamma}_{J_l}$ is the canonical mapping. We call $\hat{T}(n)$ the induced operator of $T(n)$. It is easy to see that $\hat{T}(n)$ is a bounded linear operator on $\hat{\mathcal{B}}^{\gamma}_{J_l}$, and it satisfies the semigroup property. In the following, as in [7], we will focus our attention to the quotient space $\hat{\mathcal{B}}^{\gamma}_{J_l}$ rather than $\mathcal{B}^{\gamma}_{J_l}$ to apply several results in the theory of Banach spaces. By repeating almost the same argument as in [8, Lemma 4.2], one can see that any λ belonging to the spectrum $\sigma(\hat{T})$ of $\hat{T} := \hat{T}(1)$ with $|\lambda| > 1/\gamma$ is characterized as a root of $\det \Delta(z) = 0$. Let ρ be any constant satisfying $\rho > 1/\gamma$ and $\det \Delta(z) \neq 0$ for all z with $|z| = \rho$, and consider the set

$$\Sigma_\rho := \bigl\{\lambda \in \mathbf{C} \mid \det \Delta(\lambda) = 0,\ |\lambda| > \rho\bigr\}.$$

Then Σ_ρ is a finite set because Σ_ρ does not intersect with the essential spectrum of $\hat{T}$, and therefore, the space $\hat{\mathcal{B}}^{\gamma}_{J_l}$ is decomposed as a direct sum

$$\hat{\mathcal{B}}^{\gamma}_{J_l} = \hat{U} \oplus \hat{S},$$

where $\hat{U} := \hat{U}_\rho$ and $\hat{S} := \hat{S}_\rho$ are some invariant closed subspaces of $\hat{\mathcal{B}}^{\gamma}_{J_l}$ which correspond to Σ_ρ. Hereafter, we use the notations $\hat{T}^S \equiv \hat{T}|_{\hat{S}} : \hat{S} \to \hat{S}$ and $\hat{T}^U \equiv \hat{T}|_{\hat{U}} : \hat{U} \to \hat{U}$. Note that $\sigma(\hat{T}^U) = \Sigma_\rho$ and $\sigma(\hat{T}^S) = \sigma(\hat{T}) \setminus \Sigma_\rho$. Also, there exist positive constants ν and μ with $1/\gamma < \nu < \rho < \mu$ such that

$$r(\hat{T}^S) < \nu \quad \text{and} \quad r\bigl((\hat{T}^U)^{-1}\bigr) < \frac{1}{\mu}, \tag{3.2}$$

where $r(\hat{T})$ is the spectral radius of $\hat{T}$.

Lemma 3.1. *There exists an equivalent norm $|\cdot|_{\hat{\mathcal{B}}^{\gamma}_{J_l}}$ on $\hat{\mathcal{B}}^{\gamma}_{J_l}$ such that the operator norms $|\hat{T}^S|$ and $|\hat{T}^U|$ corresponding to $|\cdot|_{\hat{\mathcal{B}}^{\gamma}_{J_l}}$ satisfy*

$$|\hat{T}^S| \leqslant \nu \quad \textit{and} \quad \bigl|(\hat{T}^U)^{-1}\bigr| \leqslant \frac{1}{\mu},$$

respectively.

Proof. For $\hat{\phi} \in \hat{\mathcal{B}}^{\gamma}_{J_l}$, let

$$|\hat{\phi}|_{\hat{\mathcal{B}}^{\gamma}_{J_l}} := \sup_{n \geqslant 0}\bigl(\bigl\|(\hat{T}^S)^n \hat{\Pi}^S \hat{\phi}\bigr\|_{\hat{\mathcal{B}}^{\gamma}_{J_l}} \nu^{-n} + \bigl\|(\hat{T}^U)^{-n} \hat{\Pi}^U \hat{\phi}\bigr\|_{\hat{\mathcal{B}}^{\gamma}_{J_l}} \mu^n\bigr),$$

where $\hat{\Pi}^S : \hat{\mathcal{B}}^{\gamma}_{J_l} \to \hat{S}$ and $\hat{\Pi}^U : \hat{\mathcal{B}}^{\gamma}_{J_l} \to \hat{U}$ denote the projection operators which correspond to the above decomposition. By virtue of (3.2) and the formula for the spectral radius $r(\hat{T}) = \lim_{n\to\infty} \sqrt[n]{\|\hat{T}^n\|}$, there exists a constant K with $K \geqslant 1$ such that

$$\bigl\|(\hat{T}^S)^n\bigr\| \leqslant K\nu^n \quad \text{and} \quad \bigl\|(\hat{T}^U)^{-n}\bigr\| \leqslant K\mu^{-n}, \quad n \geqslant 0.$$

Then it follows that for $\hat\phi \in \hat{\mathcal{B}}^\gamma_{J_l}$,

$$|\hat\phi|_{\hat{\mathcal{B}}^\gamma_{J_l}} \geqslant \|\hat\Pi^S\hat\phi\|_{\hat{\mathcal{B}}^\gamma_{J_l}} + \|\hat\Pi^U\hat\phi\|_{\hat{\mathcal{B}}^\gamma_{J_l}} \geqslant \big\|(\hat\Pi^S + \hat\Pi^U)\hat\phi\big\|_{\hat{\mathcal{B}}^\gamma_{J_l}} = \|\hat\phi\|_{\hat{\mathcal{B}}^\gamma_{J_l}}$$

and

$$\begin{aligned}|\hat\phi|_{\hat{\mathcal{B}}^\gamma_{J_l}} &\leqslant \sup_{n\geqslant 0}\big(\big\|(\hat T^S)^n\big\|\|\hat\Pi^S\|\|\hat\phi\|_{\hat{\mathcal{B}}^\gamma_{J_l}}\nu^{-n} + \big\|(\hat T^U)^{-n}\big\|\|\hat\Pi^U\|\|\hat\phi\|_{\hat{\mathcal{B}}^\gamma_{J_l}}\mu^n\big)\\ &\leqslant K\big(\|\hat\Pi^S\| + \|\hat\Pi^U\|\big)\|\hat\phi\|_{\hat{\mathcal{B}}^\gamma_{J_l}},\end{aligned}$$

which imply that

$$\|\hat\phi\|_{\hat{\mathcal{B}}^\gamma_{J_l}} \leqslant |\hat\phi|_{\hat{\mathcal{B}}^\gamma_{J_l}} \leqslant C_0\|\hat\phi\|_{\hat{\mathcal{B}}^\gamma_{J_l}} \quad \text{for } \hat\phi \in \hat{\mathcal{B}}^\gamma_{J_l}, \tag{3.3}$$

where $C_0 = K(\|\hat\Pi^S\| + \|\hat\Pi^U\|)$. Hence, the norms $\|\cdot\|_{\hat{\mathcal{B}}^\gamma_{J_l}}$ and $|\cdot|_{\hat{\mathcal{B}}^\gamma_{J_l}}$ are equivalent on $\hat{\mathcal{B}}^\gamma_{J_l}$. Then we have

$$\begin{aligned}|\hat T^S\hat\phi|_{\hat{\mathcal{B}}^\gamma_{J_l}} &= \sup_{n\geqslant 0}\big(\big\|(\hat T^S)^n\hat\Pi^S\hat T^S\hat\phi\big\|_{\hat{\mathcal{B}}^\gamma_{J_l}}\nu^{-n} + \big\|(\hat T^U)^{-n}\,\hat\Pi^U\hat T^S\hat\phi\big\|_{\hat{\mathcal{B}}^\gamma_{J_l}}\mu^n\big)\\ &= \sup_{n\geqslant 0}\big(\big\|(\hat T^S)^{n+1}\hat\Pi^S\hat\phi\big\|_{\hat{\mathcal{B}}^\gamma_{J_l}}\nu^{-(n+1)}\big)\nu \leqslant \nu|\hat\phi|_{\hat{\mathcal{B}}^\gamma_{J_l}},\end{aligned}$$

that is,

$$|\hat T^S| = \sup_{|\hat\phi|_{\hat{\mathcal{B}}^\gamma_{J_l}}\leqslant 1}|\hat T^S\hat\phi|_{\hat{\mathcal{B}}^\gamma_{J_l}} \leqslant \nu.$$

Similarly, we get

$$\begin{aligned}\big|(\hat T^U)^{-1}\hat\phi\big|_{\hat{\mathcal{B}}^\gamma_{J_l}} &= \sup_{n\geqslant 0}\big(\big\|(\hat T^S)^n\hat\Pi^S(\hat T^U)^{-1}\hat\phi\big\|_{\hat{\mathcal{B}}^\gamma_{J_l}}\nu^{-n}\\ &\qquad + \big\|(\hat T^U)^{-n}\,\hat\Pi^U(\hat T^U)^{-1}\hat\phi\big\|_{\hat{\mathcal{B}}^\gamma_{J_l}}\mu^n\big)\\ &= \sup_{n\geqslant 0}\big(\big\|(\hat T^U)^{-(n+1)}\hat\Pi^U\hat\phi\big\|_{\hat{\mathcal{B}}^\gamma_{J_l}}\mu^{n+1}\big)\frac{1}{\mu} \leqslant \frac{1}{\mu}|\hat\phi|_{\hat{\mathcal{B}}^\gamma_{J_l}},\end{aligned}$$

namely,

$$\big|(\hat T^U)^{-1}\big| = \sup_{|\hat\phi|_{\hat{\mathcal{B}}^\gamma_{J_l}}\leqslant 1}\big|(\hat T^U)^{-1}\hat\phi\big|_{\hat{\mathcal{B}}^\gamma_{J_l}} \leqslant \frac{1}{\mu}.$$

This completes the proof. □

4. Proof of main results

In the following, we will restrict our consideration to the case $\mathcal{B}^\gamma_{J_\infty} =: \mathcal{B}^\gamma$ in order to avoid several cumbersome notations such as $\hat T$, $\hat\Pi^S$ and so on, and establish Theorem 2.1.

For the proof of Theorem 2.1 with $\mathcal{B}^{\gamma}_{J_l}$ for some $l \in \mathbf{Z}^+$, one can proceed with the argument by applying Lemma 3.1 through the quotient space $\hat{\mathcal{B}}^{\gamma}_{J_l}$, the induced operator $\hat{T}(n)$ and so on which were introduced in Section 3; so, we will omit a treatment for the case of $\mathcal{B}^{\gamma}_{J_l}$ with $l \in \mathbf{Z}^+$.

The following proposition plays an essential role in the development of this section.

Proposition 4.1. *Suppose* (1.3) *holds. Let x be a solution of Eq.* (1.2) *such that*

$$\limsup_{n\to\infty} \sqrt[n]{\|x_n\|_{\mathcal{B}^\gamma}} > \frac{1}{\gamma},$$

and let ρ be any constant satisfying

$$\frac{1}{\gamma} < \rho < \limsup_{n\to\infty} \sqrt[n]{\|x_n\|_{\mathcal{B}^\gamma}} \quad \text{and} \quad \det \Delta(z) \neq 0$$

for all z with $|z| = \rho$. Then

$$\lim_{n\to\infty} \frac{\|\Pi^S x_n\|_{\mathcal{B}^\gamma}}{\|\Pi^U x_n\|_{\mathcal{B}^\gamma}} = 0,$$

where Π^S and Π^U are the projection operators corresponding to the decomposition of $\mathcal{B}^\gamma$.

To prove Proposition 4.1, we need the following lemma.

Lemma 4.1. *Suppose* (1.3) *holds. If x is a solution of Eq.* (1.2)*, then either $|x_n|_{\mathcal{B}^\gamma} = 0$ for all large $n \in \mathbf{Z}^+$ or the limit*

$$\lim_{n\to\infty} \frac{|\Pi^U x_n|_{\mathcal{B}^\gamma}}{|\Pi^U x_n|_{\mathcal{B}^\gamma} + |\Pi^S x_n|_{\mathcal{B}^\gamma}}$$

exists and is equal to 0 *or* 1. *Here, $|\cdot|_{\mathcal{B}^\gamma}$ is the norm ensured in Lemma* 3.1.

Proof. Let x be a solution of Eq. (1.2). If there exists a positive integer n_0 such that $|x_{n_0}|_{\mathcal{B}^\gamma} = 0$, then $\|x_{n_0}\|_{\mathcal{B}^\gamma} = 0$ because of $\|x_{n_0}\|_{\mathcal{B}^\gamma} \leqslant |x_{n_0}|_{\mathcal{B}^\gamma} = 0$. Hence we have

$$\left|x(n_0+1)\right| \leqslant \left|L(x_{n_0})\right| + \left|G(n_0)x_{n_0}\right| \leqslant \left(\|L\| + \left\|G(n_0)\right\|\right)\|x_{n_0}\|_{\mathcal{B}^\gamma} = 0,$$

that is, $|x(n_0+1)| = 0$, and therefore, by (3.3),

$$\begin{aligned} |x_{n_0+1}|_{\mathcal{B}^\gamma} \leqslant C_0\|x_{n_0+1}\|_{\mathcal{B}^\gamma} &= C_0 \sup_{m\geqslant 0}\left|x(n_0+1-m)\right|\gamma^{-m} \\ &= C_0 \sup_{\tau\geqslant 0}\left|x(n_0-\tau)\right|\gamma^{-\tau-1} \\ &= \frac{C_0}{\gamma}\|x_{n_0}\|_{\mathcal{B}^\gamma} = 0, \end{aligned}$$

namely, $|x_{n_0+1}|_{\mathcal{B}^\gamma} = 0$. By induction, $|x_n|_{\mathcal{B}^\gamma} = 0$ for all $n \geqslant n_0$. From now on, we will exclude this case. Thus, we may assume that $0 < |x_n|_{\mathcal{B}^\gamma} \leqslant |\Pi^U x_n|_{\mathcal{B}^\gamma} + |\Pi^S x_n|_{\mathcal{B}^\gamma}$ for $n \geqslant 0$.

For simplicity, define

$$M(n) := |\Pi^U x_n|_{\mathcal{B}^\gamma}, \qquad N(n) := |\Pi^S x_n|_{\mathcal{B}^\gamma} \tag{4.1}$$

for $n \geqslant 0$. Then, by virtue of Proposition 3.1, it follows that

$$x_{n+1} = T(1)x_n + \Gamma G(n)x_n.$$

This yields

$$\Pi x_{n+1} = \Pi T x_n + \Pi \big(\Gamma G(n)x_n\big), \quad \Pi = \Pi^S \text{ or } \Pi^U, \tag{4.2}$$

and hence, by Lemma 3.1, we have

$$\begin{aligned} N(n+1) = |\Pi^S x_{n+1}|_{\mathcal{B}^\gamma} &\leqslant |T^S||\Pi^S x_n|_{\mathcal{B}^\gamma} + |\Pi^S|\big|\Gamma G(n)\big|\big|(\Pi^S + \Pi^U)x_n\big|_{\mathcal{B}^\gamma} \\ &\leqslant \nu N(n) + |\Pi^S|\big|\Gamma G(n)\big|\big(N(n) + M(n)\big), \end{aligned}$$

namely,

$$N(n+1) \leqslant \nu N(n) + \beta(n)\big(N(n) + M(n)\big) \quad \text{for } n \geqslant 0, \tag{4.3}$$

where $\beta(n) = (|\Pi^S| + |\Pi^U|)|\Gamma G(n)|$. Note that

$$\lim_{n\to\infty} \beta(n) = 0 \tag{4.4}$$

because of (1.3) and the equivalence of the norms $\|\cdot\|_{\mathcal{B}^\gamma}$ and $|\cdot|_{\mathcal{B}^\gamma}$ (Lemma 3.1). Also, by (4.2), we get

$$\begin{aligned} M(n+1) = |\Pi^U x_{n+1}|_{\mathcal{B}^\gamma} &\geqslant |T^U \Pi^U x_n|_{\mathcal{B}^\gamma} - |\Pi^U|\big|\Gamma G(n)\big|\big|(\Pi^S + \Pi^U)x_n\big|_{\mathcal{B}^\gamma} \\ &\geqslant |T^U \Pi^U x_n|_{\mathcal{B}^\gamma} - \beta(n)\big(N(n) + M(n)\big). \end{aligned}$$

Since $|y|_{\mathcal{B}^\gamma} = |(T^U)^{-1}T^U y|_{\mathcal{B}^\gamma} \leqslant |(T^U)^{-1}||T^U y|_{\mathcal{B}^\gamma} \leqslant (1/\mu)|T^U y|_{\mathcal{B}^\gamma}$ by Lemma 3.1, we obtain

$$\mu|y|_{\mathcal{B}^\gamma} \leqslant |T^U y|_{\mathcal{B}^\gamma} \quad \text{for } y \in U,$$

which, together with the above relation, implies

$$M(n+1) \geqslant \mu M(n) - \beta(n)\big(N(n) + M(n)\big) \quad \text{for } n \geqslant 0. \tag{4.5}$$

Suppose that

$$\lim_{n\to\infty} \frac{M(n)}{M(n) + N(n)} = 0 \tag{4.6}$$

does not hold. Then there exists $\varepsilon_0 \in (0, 1)$ such that

$$\frac{M(n)}{M(n) + N(n)} \geqslant \varepsilon_0 \quad \text{for infinitely many } n,$$

that is,

$$(1 - \varepsilon_0)M(n) \geqslant \varepsilon_0 N(n) \quad \text{for infinitely many } n. \tag{4.7}$$

By (4.4), there exists $n_1 \geqslant 0$ such that for $n \geqslant n_1$,

$$\begin{cases} \mu - \frac{\beta(n)}{\varepsilon_0} > 0, \\ \frac{(1-\varepsilon_0)\nu+\beta(n)}{\varepsilon_0\mu-\beta(n)} < \frac{1-\varepsilon_0}{\varepsilon_0}. \end{cases} \tag{4.8}$$

In fact, the second inequality of (4.8) is equivalent to $\beta(n) < \varepsilon_0(1-\varepsilon_0)(\mu-\nu)$, so the above assertion is valid. By (4.7), there exists $n_2 \geqslant n_1$ such that

$$(1-\varepsilon_0)M(n_2) \geqslant \varepsilon_0 N(n_2).$$

Claim 1.

$$(1-\varepsilon_0)M(n) \geqslant \varepsilon_0 N(n) \quad \textit{for } n \geqslant n_2.$$

Suppose for induction that this inequality holds for some $n \geqslant n_2$. Then it follows from (4.3) and (4.5) that

$$\begin{aligned} N(n+1) &\leqslant \nu N(n) + \beta(n)\big(N(n)+M(n)\big) \\ &\leqslant \left[\big(\nu+\beta(n)\big)\frac{1-\varepsilon_0}{\varepsilon_0} + \beta(n)\right]M(n) \\ &= \left[\frac{(1-\varepsilon_0)\nu}{\varepsilon_0} + \frac{\beta(n)}{\varepsilon_0}\right]M(n) \end{aligned}$$

and

$$\begin{aligned} M(n+1) &\geqslant \mu M(n) - \beta(n)\big(N(n)+M(n)\big) \\ &\geqslant \left[\mu - \beta(n)\frac{1-\varepsilon_0}{\varepsilon_0} - \beta(n)\right]M(n) \\ &= \left[\mu - \frac{\beta(n)}{\varepsilon_0}\right]M(n). \end{aligned} \tag{4.9}$$

These relations, together with (4.8), imply that

$$\begin{aligned} N(n+1) &\leqslant \frac{(1-\varepsilon_0)\nu/\varepsilon_0 + \beta(n)/\varepsilon_0}{\mu - \beta(n)/\varepsilon_0} M(n+1) \\ &= \frac{(1-\varepsilon_0)\nu + \beta(n)}{\varepsilon_0\mu - \beta(n)} M(n+1) \\ &\leqslant \frac{1-\varepsilon_0}{\varepsilon_0} M(n+1), \end{aligned}$$

and hence, the claim is verified.

From Claim 1, we have for $n \geqslant n_2$,

$$0 < |x_n| \leqslant N(n) + M(n) \leqslant \frac{1-\varepsilon_0}{\varepsilon_0}M(n) + M(n) = \frac{1}{\varepsilon_0}M(n),$$

which yields that $M(n) > 0$ for $n \geqslant n_2$. Define

$$w(n) := \frac{N(n)}{M(n)} \quad \text{for } n \geqslant n_2.$$

Then, by Claim 1 again, we get

$$\ell := \limsup_{n\to\infty} w(n) \leqslant \frac{1-\varepsilon_0}{\varepsilon_0} < \infty.$$

Moreover, (4.3) and (4.9) imply for $n \geqslant n_2$,

$$\begin{aligned} w(n+1) = \frac{N(n+1)}{M(n+1)} &\leqslant \frac{\nu N(n) + \beta(n)(N(n)+M(n))}{(\mu - \beta(n)/\varepsilon_0)M(n)} \\ &\leqslant \frac{\nu + \beta(n)}{\mu - \beta(n)/\varepsilon_0} w(n) + \frac{\beta(n)}{\mu - \beta(n)/\varepsilon_0}. \end{aligned}$$

Taking the lim sup on the both sides of the last inequality and using (4.4), we obtain $\ell \leqslant (\nu/\mu)\ell$. Since $\nu < \mu$, we have $\ell = 0$ and thus $\lim_{n\to\infty} w(n)$ exists and is zero. We therefore conclude that if (4.6) does not hold, then

$$\lim_{n\to\infty} \frac{M(n)}{M(n)+N(n)} = \lim_{n\to\infty} \frac{1}{1+w(n)} = 1.$$

This completes the proof of the lemma. □

Proof of Proposition 4.1. Let x be a solution of Eq. (1.2) such that

$$\limsup_{n\to\infty} \sqrt[n]{\|x_n\|_{\mathcal{B}^\gamma}} > \frac{1}{\gamma}.$$

Let $|\cdot|_{\mathcal{B}^\gamma}$ be the one ensured in Lemma 3.1. We first note that

$$\limsup_{n\to\infty} \sqrt[n]{|x_n|_{\mathcal{B}^\gamma}} = \limsup_{n\to\infty} \sqrt[n]{\|x_n\|_{\mathcal{B}^\gamma}} > \frac{1}{\gamma}.$$

Also, using (3.3), we have

$$\|\Pi^S x_n\|_{\mathcal{B}^\gamma} \leqslant |\Pi^S x_n|_{\mathcal{B}^\gamma} = \frac{|\Pi^S x_n|_{\mathcal{B}^\gamma}}{|\Pi^U x_n|_{\mathcal{B}^\gamma}} |\Pi^U x_n|_{\mathcal{B}^\gamma} \leqslant C_0 \frac{|\Pi^S x_n|_{\mathcal{B}^\gamma}}{|\Pi^U x_n|_{\mathcal{B}^\gamma}} \|\Pi^U x_n\|_{\mathcal{B}^\gamma},$$

which yields

$$\frac{\|\Pi^S x_n\|_{\mathcal{B}^\gamma}}{\|\Pi^U x_n\|_{\mathcal{B}^\gamma}} \leqslant C_0 \frac{|\Pi^S x_n|_{\mathcal{B}^\gamma}}{|\Pi^U x_n|_{\mathcal{B}^\gamma}} \equiv C_0 \frac{N(n)}{M(n)},$$

where $M(n)$ and $N(n)$ are defined as (4.1). Hence, if we show that

$$\lim_{n\to\infty} \frac{N(n)}{M(n)} = 0, \tag{4.10}$$

then the proof will be complete.

Claim 2.

$$\lim_{n\to\infty} \frac{M(n)}{M(n)+N(n)} \neq 0.$$

Suppose that the claim is not true. Since the limit $\lim_{n\to\infty} M(n)/(M(n)+N(n))$ exists by Lemma 4.1, we must get

$$\lim_{n\to\infty} \frac{M(n)}{M(n)+N(n)} = 0.$$

Then there exists a sufficiently large integer $n_1 \geqslant 0$ such that $M(n)/(M(n)+N(n)) \leqslant 1/2$ for $n \geqslant n_1$, namely,

$$M(n) \leqslant N(n) \quad \text{for } n \geqslant n_1,$$

and hence, the inequality (4.3), together with $\nu < \rho$, yields

$$N(n+1) \leqslant \bigl(\rho + 2\beta(n)\bigr)N(n) \quad \text{for } n \geqslant n_1.$$

Let θ be a positive constant satisfying

$$\rho < \theta < \limsup_{n\to\infty} \sqrt[n]{|x_n|_{\mathcal{B}^\gamma}}.$$

From this and (4.4), there exists $n_2 \geqslant n_1$ such that $\beta(n) < (\theta - \rho)/2$ for $n \geqslant n_2$, that is,

$$\rho + 2\beta(n) < \theta \quad \text{for } n \geqslant n_2.$$

Since $N(n+1) \leqslant \theta N(n)$ for $n \geqslant n_2$, we get

$$N(n) \leqslant N(n_2)\theta^{n-n_2} \quad \text{for } n \geqslant n_2,$$

which implies that

$$|x_n|_{\mathcal{B}^\gamma} \leqslant M(n) + N(n) \leqslant 2N(n) \leqslant 2N(n_2)\theta^{n-n_2} \quad \text{for } n \geqslant n_2.$$

Hence

$$\limsup_{n\to\infty} \sqrt[n]{|x_n|_{\mathcal{B}^\gamma}} \leqslant \limsup_{n\to\infty} \sqrt[n]{2N(n_2)\theta^{n-n_2}} = \theta,$$

which contradicts the definition of θ and so the claim is verified.

By virtue of Lemma 4.1 and Claim 2, we have

$$\lim_{n\to\infty} \frac{M(n)}{M(n)+N(n)} = 1,$$

and thus

$$\lim_{n\to\infty} \frac{N(n)}{M(n)} = \lim_{n\to\infty} \left(\frac{M(n)+N(n)}{M(n)} - 1 \right) = 0.$$

We therefore obtain (4.10) and the proof of Proposition 4.1 is complete. □

Now, we are in a position to prove Theorem 2.1.

Proof of Theorem 2.1. Let x be a solution of Eq. (1.2) such that

$$\limsup_{n\to\infty} \sqrt[n]{\|x_n\|_{\mathcal{B}^\gamma}} > \frac{1}{\gamma},$$

and let ρ be any constant satisfying

$$\frac{1}{\gamma} < \rho < \limsup_{n\to\infty} \sqrt[n]{\|x_n\|_{\mathcal{B}^\gamma}} \quad \text{and} \quad \det \Delta(z) \neq 0$$

for all z with $|z| = \rho$. Then it follows that

$$\|\Pi^U x_n\|_{\mathcal{B}^\gamma}\left(1 - \frac{\|\Pi^S x_n\|_{\mathcal{B}^\gamma}}{\|\Pi^U x_n\|_{\mathcal{B}^\gamma}}\right) \leqslant \|x_n\|_{\mathcal{B}^\gamma} \leqslant \|\Pi^U x_n\|_{\mathcal{B}^\gamma}\left(1 + \frac{\|\Pi^S x_n\|_{\mathcal{B}^\gamma}}{\|\Pi^U x_n\|_{\mathcal{B}^\gamma}}\right).$$

By virtue of Proposition 4.1, we obtain

$$\lim_{n\to\infty} \sqrt[n]{1 \pm \frac{\|\Pi^S x_n\|_{\mathcal{B}^\gamma}}{\|\Pi^U x_n\|_{\mathcal{B}^\gamma}}} = 1,$$

and hence,

$$\lim_{n\to\infty} \sqrt[n]{\|x_n\|_{\mathcal{B}^\gamma}} = \lim_{n\to\infty} \sqrt[n]{\|\Pi^U x_n\|_{\mathcal{B}^\gamma}}.$$

Consequently, to prove Theorem 2.1, we have only to show that

$$\lim_{n\to\infty} \sqrt[n]{\|\Pi^U x_n\|_{\mathcal{B}^\gamma}} = |\lambda|, \tag{4.11}$$

where λ is a root of $\det \Delta(\lambda) = 0$ with $|\lambda| > 1/\gamma$.

To this end, we will consider the asymptotic behavior of the solution x on U. Recall that $\dim U =: d < \infty$ because the set

$$\sigma\big(T^U(1)\big) = \Sigma_\rho = \big\{\lambda \in \mathbf{C} \mid \det \Delta(\lambda) = 0,\ |\lambda| > \rho\big\}$$

is finite. Take a basis $\{\phi_1, \ldots, \phi_d\}$ in U, and set $\Phi = (\phi_1, \ldots, \phi_d)$. For any $\Phi = (\phi_1, \ldots, \phi_d)$ in U, there exists a unique family $\{\psi_1, \ldots, \psi_d\}$ in $(\mathcal{B}^\gamma)^*$ (the dual space of $\mathcal{B}^\gamma$) which satisfies the relation $\langle \psi_i, \phi_j \rangle = \delta_{ij}$, and $\psi_i \equiv 0$ on S. Here and hereafter, $\langle \cdot, \cdot \rangle$ denotes the canonical pairing between the dual space and the original space. Denote by Ψ the transpose of $(\psi_1, \ldots, \psi_d)$ to use the expression $\langle \Psi, \Phi \rangle = I_d$ (here I_d is the $d \times d$ identity matrix). Then one can easily see that the projection operator Π^U is given by

$$\Pi^U \phi = \Phi \langle \Psi, \phi \rangle, \quad \phi \in \mathcal{B}^\gamma.$$

Moreover, since $T\Phi = (T\phi_1, \ldots, T\phi_d) \in U \times \cdots \times U$, there exists a $d \times d$ complex matrix A such that

$$T\Phi = \Phi A \quad \text{and} \quad \sigma(A) = \sigma\big(T^U(1)\big) = \Sigma_\rho.$$

Let $y(n)$ be a function defined by

$$\Phi y(n) = \Pi^U x_n \quad \text{or} \quad y(n) = \langle \Psi, x_n \rangle$$

for $n \geqslant 0$. Then, from (4.2) and the above relations, it follows that

$$\begin{aligned}
\Phi y(n+1) = \Pi^U x_{n+1} &= \Pi^U \big(T x_n + \Gamma G(n) x_n\big) \\
&= T^U \Phi \langle \Psi, x_n \rangle + \Phi \big\langle \Psi, \Gamma G(n) x_n \big\rangle \\
&= \Phi A y(n) + \Phi \big\langle \Psi, \Gamma G(n) x_n \big\rangle.
\end{aligned}$$

Hence, the function y is a solution of the ordinary difference equation

$$y(n+1) = Ay(n) + \langle \Psi, \Gamma G(n)x_n \rangle. \tag{4.12}$$

In the following, we will investigate the asymptotic behavior of solutions of Eq. (4.12) by modifying a method employed in the second proof of [10, Theorem 1].

Since all norms on a finite dimensional vector space are equivalent, the existence and the value of the limit of $\sqrt[n]{|y(n)|}$ as $n \to \infty$ are independent of the norm used and are invariant under a constant invertible linear transformation. Consequently, we can assume that A is in Jordan's canonical form. For any $\alpha > 0$, set

$$P^{(\alpha)} y(n) = z^{(\alpha)}(n) \; \big(=: z(n)\big),$$

where $P^{(\alpha)} = \operatorname{diag}(1, \alpha^{-1}, \ldots, \alpha^{-d+1})$. Later, we will consider a limit of some quantity as $\alpha \to 0$. In what follows, in order to proceed with our arguments precisely, we will use the notation such as $P^{(\alpha)}$, indicating the dependence on α explicitly. Eq. (4.12) can be transformed into

$$z(n+1) = P^{(\alpha)} A (P^{(\alpha)})^{-1} z(n) + P^{(\alpha)} \langle \Psi, \Gamma G(n)x_n \rangle,$$

namely,

$$z_i(n+1) = \lambda_i z_i(n) + \alpha_i z_{i+1}(n) + b_i^{(\alpha)}(n), \quad i = 1, \ldots, d, \tag{4.13}$$

where $b^{(\alpha)}(n) = P^{(\alpha)} \langle \Psi, \Gamma G(n)x_n \rangle$ and $\lambda_1, \ldots, \lambda_d$ are the eigenvalues of A ordered so that $|\lambda_1| \geqslant |\lambda_2| \geqslant \cdots \geqslant |\lambda_d|$, $\alpha_i = 0$ or α and $\alpha_i = 0$ if $\lambda_i \neq \lambda_{i+1}$. (For $i = d$ the second term on the right-hand side is zero.)

From now on, let $\|\cdot\|$ be the l_1-norm on $\mathbf{C}^d$, that is, $\|z\| = \sum_{i=1}^d |z_i|$ for $z \in \mathbf{C}^d$. Then $\|A\| = \max_{1 \leqslant j \leqslant d} \sum_{i=1}^d |A_{ij}|$. By virtue of Proposition 4.1, there exists a sufficiently large integer $n_0 \geqslant 0$ such that

$$\|\Pi^S x_n\|_{\mathcal{B}^\gamma} \leqslant \|\Pi^U x_n\|_{\mathcal{B}^\gamma} \quad \text{for } n \geqslant n_0,$$

and hence, we have for $n \geqslant n_0$,

$$\begin{aligned}
\big\|b^{(\alpha)}(n)\big\| &\leqslant \|P^{(\alpha)}\| \, \|\Psi\| \, \big\|G(n)\big\| \, \big\|(\Pi^U + \Pi^S)x_n\big\|_{\mathcal{B}^\gamma} \\
&\leqslant 2\|P^{(\alpha)}\| \, \|\Psi\| \, \big\|G(n)\big\| \, \|\Pi^U x_n\|_{\mathcal{B}^\gamma} \\
&= 2\|P^{(\alpha)}\| \, \|\Psi\| \, \big\|G(n)\big\| \, \big\|\Phi y(n)\big\|_{\mathcal{B}^\gamma} \\
&\leqslant 2\|P^{(\alpha)}\| \, \|\Psi\| \, \big\|G(n)\big\| \, \|\Phi\| \, \big\|(P^{(\alpha)})^{-1}\big\| \, \big\|z(n)\big\| \\
&= K^{(\alpha)} \big\|G(n)\big\| \, \big\|z(n)\big\|,
\end{aligned} \tag{4.14}$$

where $\|\Psi\| = \max\{\|\psi_1\|_{(\mathcal{B}^\gamma)^*}, \ldots, \|\psi_d\|_{(\mathcal{B}^\gamma)^*}\}$, $\|\Phi\| = \max\{\|\phi_1\|_{\mathcal{B}^\gamma}, \ldots, \|\phi_d\|_{\mathcal{B}^\gamma}\}$, and $K^{(\alpha)} = 2\|P^{(\alpha)}\| \, \|\Psi\| \|\Phi\| \|(P^{(\alpha)})^{-1}\|$.

Let $\rho_1 > \rho_2 > \cdots > \rho_h$ be the distinct moduli of the eigenvalues of A and suppose that α is chosen so small that

$$2\alpha < \rho_m - \rho_{m+1}, \quad m = 1, \ldots, h-1.$$

For brevity, define

$$L_m^{(\alpha)}(n) := \sum_{|\lambda_i|=\rho_m} \left|z_i^{(\alpha)}(n)\right|,$$

$$M_m^{(\alpha)}(n) := \sum_{|\lambda_i|>\rho_m} \left|z_i^{(\alpha)}(n)\right|,$$

$$N_m^{(\alpha)}(n) := \sum_{|\lambda_i|\leqslant\rho_m} \left|z_i^{(\alpha)}(n)\right|$$

for $m = 1, \ldots, h$ and $n \geqslant n_0$. (It is assumed that the solution x of (1.2) is defined for $n \geqslant n_0$.) Clearly, $M_m^{(\alpha)}(n) + N_m^{(\alpha)}(n) = \|z^{(\alpha)}(n)\|$. From (4.13), (4.14) and the fact that $\alpha_i = 0$ if $\lambda_i \neq \lambda_{i+1}$, we get for $m = 1, \ldots, h$ and $n \geqslant n_0$,

$$\left|L_m^{(\alpha)}(n+1) - \rho_m L_m^{(\alpha)}(n)\right| \leqslant \alpha L_m^{(\alpha)}(n) + K^{(\alpha)}\left\|G(n)\right\|\left\|z^{(\alpha)}(n)\right\|, \tag{4.15}$$

$$M_m^{(\alpha)}(n+1) \geqslant (\rho_{m-1} - \alpha) M_m^{(\alpha)}(n) - K^{(\alpha)}\left\|G(n)\right\|\left\|z^{(\alpha)}(n)\right\|, \tag{4.16}$$

$$N_m^{(\alpha)}(n+1) \leqslant (\rho_m + \alpha) N_m^{(\alpha)}(n) + K^{(\alpha)}\left\|G(n)\right\|\left\|z^{(\alpha)}(n)\right\|. \tag{4.17}$$

If $z^{(\alpha)}(n_1) = 0$ for some $n_1 \geqslant n_0$, then $z^{(\alpha)}(n) = 0$ for all $n \geqslant n_1$. From now on, we exclude this case. Thus we may assume that $\|z^{(\alpha)}(n)\| = M_m^{(\alpha)}(n) + N_m^{(\alpha)}(n) > 0$ for $n \geqslant n_0$.

Now, using the inequalities (4.16) and (4.17) instead of (4.3) and (4.5), we can obtain the following lemma in a similar way to Lemma 4.1 (we here omit the proof).

Lemma 4.2. *Suppose* (1.3) *holds. Then for any $m = 1, \ldots, h$, the limit*

$$\lim_{n\to\infty} \frac{M_m^{(\alpha)}(n)}{M_m^{(\alpha)}(n) + N_m^{(\alpha)}(n)}$$

exists and is equal to 0 *or* 1.

For any $m = 1, \ldots, h$ and $n \geqslant n_0$, define

$$v_m^{(\alpha)}(n) := \frac{M_m^{(\alpha)}(n)}{\|z^{(\alpha)}(n)\|} = \frac{M_m^{(\alpha)}(n)}{M_m^{(\alpha)}(n) + N_m^{(\alpha)}(n)}.$$

Note that $v_1^{(\alpha)}(n)$ tends to zero as $n \to \infty$, because $M_1^{(\alpha)}(n) \equiv 0$. Let $g^{(\alpha)}$ be the greatest value of m for which $v_m^{(\alpha)}(n)$ tends to zero as $n \to \infty$. Since $M_m^{(\alpha)}(n) = L_1^{(\alpha)}(n) + \cdots + L_{m-1}^{(\alpha)}(n)$, it follows from Lemma 4.2 and the definition of $g^{(\alpha)}$ that

$$\lim_{n\to\infty} v_m^{(\alpha)}(n) = \lim_{n\to\infty} \frac{L_1^{(\alpha)}(n) + \cdots + L_{m-1}^{(\alpha)}(n)}{\|z^{(\alpha)}(n)\|} = \begin{cases} 0 & \text{for } 1 \leqslant m \leqslant g^{(\alpha)}, \\ 1 & \text{for } g^{(\alpha)} + 1 \leqslant m \leqslant h. \end{cases}$$

Consequently,

$$\lim_{n\to\infty} \frac{L_m^{(\alpha)}(n)}{\|z^{(\alpha)}(n)\|} = \lim_{n\to\infty} \left(v_{m+1}^{(\alpha)}(n) - v_m^{(\alpha)}(n)\right) = \begin{cases} 0 & \text{for } m \neq g^{(\alpha)}, \\ 1 & \text{for } m = g^{(\alpha)}, \end{cases} \tag{4.18}$$

and thus

$$\lim_{n\to\infty}\frac{L_m^{(\alpha)}(n)}{L_{g^{(\alpha)}}^{(\alpha)}(n)}=\lim_{n\to\infty}\frac{L_m^{(\alpha)}(n)/\|z^{(\alpha)}(n)\|}{L_{g^{(\alpha)}}^{(\alpha)}(n)/\|z^{(\alpha)}(n)\|}=0\quad\text{for } m\neq g^{(\alpha)}. \tag{4.19}$$

Also, letting $m=g^{(\alpha)}$ in (4.15), we have

$$\left|\frac{L_{g^{(\alpha)}}^{(\alpha)}(n+1)}{L_{g^{(\alpha)}}^{(\alpha)}(n)}-\rho_{g^{(\alpha)}}\right|\leqslant\alpha+K^{(\alpha)}\|G(n)\|\frac{1}{L_{g^{(\alpha)}}^{(\alpha)}(n)/\|z^{(\alpha)}(n)\|}.$$

From (1.3) and (4.18), there exists $n_2^{(\alpha)}\geqslant n_0$ such that

$$\rho_{g^{(\alpha)}}-2\alpha\leqslant\frac{L_{g^{(\alpha)}}^{(\alpha)}(n+1)}{L_{g^{(\alpha)}}^{(\alpha)}(n)}\leqslant\rho_{g^{(\alpha)}}+2\alpha\quad\text{for } n\geqslant n_2^{(\alpha)},$$

which implies

$$L_{g^{(\alpha)}}^{(\alpha)}(n)(\rho_{g^{(\alpha)}}-2\alpha)\leqslant L_{g^{(\alpha)}}^{(\alpha)}(n+1)\leqslant L_{g^{(\alpha)}}^{(\alpha)}(n)(\rho_{g^{(\alpha)}}+2\alpha)\quad\text{for } n\geqslant n_2^{(\alpha)}. \tag{4.20}$$

Since $g^{(\alpha)}\in\{1,\dots,h\}$, $g^{(\alpha)}$ is a constant for infinitely many α with $\alpha\to+0$. From now on, restricting our consideration to such numbers α, we may assume $g^{(\alpha)}=g$; a constant, where $\alpha\to+0$. By the definition of $z^{(\alpha)}(n)$, it follows that

$$L_g^{(\alpha)}(n)=\sum_{|\lambda_i|=\rho_g}\left|z_i^{(\alpha)}(n)\right|=\sum_{|\lambda_i|=\rho_g}\alpha^{-i+1}\left|y_i(n)\right|.$$

Note that $L_g^{(\alpha)}(n)$ and $\|z^{(\alpha)}(n)\|$ depend on α. Define

$$L_g^*(n):=\sum_{|\lambda_i|=\rho_g}\left|y_i(n)\right|.$$

Since $\alpha>0$ is sufficiently small, we have

$$L_g^*(n)\leqslant L_g^{(\alpha)}(n)\leqslant\alpha^{-d+1}L_g^*(n)\quad\text{for } n\geqslant n_0,$$

which, together with (4.20), yields

$$C_1^{(\alpha)}(\rho_g-2\alpha)^n\leqslant L_g^*(n)\leqslant C_2^{(\alpha)}(\rho_g+2\alpha)^n\quad\text{for } n\geqslant n_2^{(\alpha)},$$

where $C_1^{(\alpha)}=\alpha^{d-1}L_g(n_2^{(\alpha)})(\rho_g-2\alpha)^{-n_2^{(\alpha)}}$ and $C_2^{(\alpha)}=L_g(n_2^{(\alpha)})(\rho_g+2\alpha)^{-n_2^{(\alpha)}}$. Thus,

$$\rho_g-2\alpha\leqslant\liminf_{n\to\infty}\sqrt[n]{L_g^*(n)}\leqslant\limsup_{n\to\infty}\sqrt[n]{L_g^*(n)}\leqslant\rho_g+2\alpha,$$

and hence, letting $\alpha\to+0$, we obtain

$$\lim_{n\to\infty}\sqrt[n]{L_g^*(n)}=\rho_g \tag{4.21}$$

because $L_g^*(n)$ is independent of α. Moreover, by (4.19), we have for $m\neq g$,

$$\frac{\sum_{|\lambda_j|=\rho_m}|y_j(n)|}{\sum_{|\lambda_i|=\rho_g}|y_i(n)|} \leqslant \frac{\sum_{|\lambda_j|=\rho_m}|y_j(n)|\alpha^{1-j}}{(\sum_{|\lambda_i|=\rho_g}|y_i(n)|\alpha^{1-i})\alpha^{d-1}} = \frac{1}{\alpha^{d-1}}\frac{L_m^{(\alpha)}(n)}{L_g^{(\alpha)}(n)} \to 0$$
$$\text{as } n \to \infty,$$

namely,

$$\lim_{n\to\infty}\frac{\sum_{|\lambda_j|=\rho_m}|y_j(n)|}{\sum_{|\lambda_i|=\rho_g}|y_i(n)|} = 0.$$

Therefore

$$\lim_{n\to\infty}\frac{\|y(n)\|}{L_g^*(n)} = \lim_{n\to\infty}\left(\frac{\sum_{m\neq g}\sum_{|\lambda_j|=\rho_m}|y_j(n)|}{\sum_{|\lambda_i|=\rho_g}|y_i(n)|}+1\right) = 1,$$

which, together with (4.21), implies

$$\lim_{n\to\infty}\sqrt[n]{\|y(n)\|} = \lim_{n\to\infty}\frac{1}{\sqrt[n]{L_g^*(n)/\|y(n)\|}}\sqrt[n]{L_g^*(n)} = \rho_g.$$

To be precise, there exists $\lambda^* \in \sigma(A) = \Sigma_\rho$ such that $\lim_{n\to\infty}\sqrt[n]{\|y(n)\|} = |\lambda^*|$. Since $\|y\| = \|\langle\Psi,\Phi\rangle y\| = \|\Psi(\Phi y)\| \leqslant \|\Psi\|\|\Phi y\|_{\mathcal{B}^\gamma}$ for $y \in \mathbf{C}^d$, we have $\|\Psi\|^{-1}\|y\| \leqslant \|\Phi y\|_{\mathcal{B}^\gamma}$. By using the above inequality and $\|\Phi y\|_{\mathcal{B}^\gamma} \leqslant \|\Phi\|\|y\|$, we get

$$\|\Psi\|^{-1}\|y(n)\| \leqslant \|\Phi y(n)\|_{\mathcal{B}^\gamma} \leqslant \|\Phi\|\|y(n)\|.$$

This yields

$$\lim_{n\to\infty}\sqrt[n]{\|\Phi y(n)\|_{\mathcal{B}^\gamma}} = \lim_{n\to\infty}\sqrt[n]{\|y(n)\|} = |\lambda^*|,$$

and hence,

$$\lim_{n\to\infty}\sqrt[n]{\|\Pi^U x_n\|_{\mathcal{B}^\gamma}} = \lim_{n\to\infty}\sqrt[n]{\|\Phi y(n)\|_{\mathcal{B}^\gamma}} = |\lambda^*|.$$

We thus obtain (4.11) and the proof of Theorem 2.1 is now complete. □

Proof of Theorem 2.2. Let x be a solution of Eq. (1.2) such that

$$\limsup_{n\to\infty}\sqrt[n]{|x(n)|} > \frac{1}{\gamma}.$$

Then we have

$$\limsup_{n\to\infty}\sqrt[n]{\|x_n\|_{\mathcal{B}^\gamma_{J_l}}} \geqslant \limsup_{n\to\infty}\sqrt[n]{|x(n)|} > \frac{1}{\gamma}.$$

By virtue of Theorem 2.1, there exists a root λ of $\det\Delta(\lambda) = 0$ with $|\lambda| > 1/\gamma$ such that

$$\lim_{n\to\infty}\sqrt[n]{\|x_n\|_{\mathcal{B}^\gamma_{J_l}}} = |\lambda|,$$

and therefore, we obtain

$$\limsup_{n\to\infty}\sqrt[n]{|x(n)|} \leqslant \limsup_{n\to\infty}\sqrt[n]{\|x_n\|_{\mathcal{B}^\gamma_{J_l}}} = |\lambda|.$$

To end the proof, we will show that

$$\limsup_{n\to\infty} \sqrt[n]{|x(n)|} \geqslant |\lambda|. \tag{4.22}$$

Without loss of generality, we may assume that x is a solution of Eq. (1.2) on $\mathbf{Z}^+$; that is, $x_0 \in \mathcal{B}^\gamma_{J_l}$ and x satisfies Eq. (1.2) on $\mathbf{Z}^+$. Recall that $\gamma \geqslant 1$. By virtue of the definition of the norm $\|\cdot\|_{\mathcal{B}^\gamma}$, one can certify that

$$\|x_n\|_{\mathcal{B}^\gamma_{J_l}} \leqslant \max\Big\{\sup_{n-\tau\leqslant s\leqslant n} |x(s)|,\ \gamma^{-\tau}\|x_{n-\tau}\|_{\mathcal{B}^\gamma_{J_l}}\Big\} \quad \text{for } \tau = 1,\dots,n-1. \tag{4.23}$$

For any rational number $c \in (0,1)$, let m be a positive integer with $cm \in \mathbf{Z}^+$. If there exist infinitely many m such that

$$\sup_{m-cm\leqslant s\leqslant m} |x(s)| \leqslant \gamma^{-cm}\|x_{m-cm}\|_{\mathcal{B}^\gamma_{J_l}},$$

then (4.23) becomes

$$\sqrt[m]{\|x_m\|_{\mathcal{B}^\gamma_{J_l}}} \leqslant \sqrt[m]{\gamma^{-cm}\|x_{m-cm}\|_{\mathcal{B}^\gamma_{J_l}}} = \gamma^{-c}\Big(\sqrt[m(1-c)]{\|x_{m(1-c)}\|_{\mathcal{B}^\gamma_{J_l}}}\Big)^{1-c}.$$

Letting $m \to \infty$, we have

$$|\lambda| \leqslant \gamma^{-c}|\lambda|^{1-c} < |\lambda|^c \cdot |\lambda|^{1-c} = |\lambda|,$$

which is a contradiction. Hence we must get

$$\sqrt[m]{\|x_m\|_{\mathcal{B}^\gamma_{J_l}}} \leqslant \sqrt[m]{\max_{m(1-c)\leqslant s\leqslant m} |x(s)|} \tag{4.24}$$

except for finitely many integer m with $cm \in \mathbf{Z}^+$.

Now let $\{\varepsilon_j\}$, $0 < \varepsilon_j < |\lambda|$, be any sequence tending to zero as $j \to \infty$. For $c = 2^{-j}$, we choose a sequence $\{m_j\} \subset \mathbf{Z}^+$ such that the inequality (4.24) holds for $m = m_j$. Without loss of generality, we may assume that $\{m_j\}$ is a strictly increasing sequence tending to infinity as $j \to \infty$, and that

$$|\lambda| - \varepsilon_j < \sqrt[m_j]{\|x_{m_j}\|_{\mathcal{B}^\gamma_{J_l}}} \quad \text{for } j = 1,2,\dots.$$

Then there exists a sequence $\{s_j\} \subset \mathbf{Z}^+$ such that $m_j(1-2^{-j}) \leqslant s_j \leqslant m_j$ and

$$|\lambda| - \varepsilon_j < \sqrt[m_j]{|x(s_j)|} \quad \text{for } j = 1,2,\dots.$$

Using the fact that

$$\lim_{j\to\infty} s_j = \infty \quad \text{and} \quad \lim_{j\to\infty} \frac{m_j}{s_j} = 1,$$

we get

$$\limsup_{j\to\infty} \sqrt[s_j]{|x(s_j)|} \geqslant \limsup_{j\to\infty} \big(|\lambda| - \varepsilon_j\big)^{m_j/s_j} = |\lambda|,$$

and therefore, (4.22) is verified. This completes the proof of Theorem 2.2. □

References

[1] S. Castillo, M. Pinto, Asymptotic formulae for nonlinear functional difference equations, Advances in difference equations, III, Comput. Math. Appl. 42 (2001) 551–559.

[2] C. Cuevas, M. Pinto, Asymptotic properties of solutions to nonautonomous Volterra difference systems with infinite delay, Advances in difference equations, III, Comput. Math. Appl. 42 (2001) 671–685.

[3] C. Cuevas, M. Pinto, Convergent solutions of linear functional difference equations in phase space, J. Math. Anal. Appl. 277 (2003) 324–341.

[4] C. Cuevas, C. Vidal, Discrete dichotomies and asymptotic behavior for abstract retarded functional difference equations in phase space, J. Differ. Equations Appl. 8 (2002) 603–640.

[5] S. Elaydi, An Introduction to Difference Equations, second ed., Springer-Verlag, New York, 1999.

[6] S. Elaydi, S. Murakami, E. Kamiyama, Asymptotic equivalence for difference equations with infinite delay, J. Differ. Equations Appl. 5 (1999) 1–23.

[7] Y. Hino, S. Murakami, T. Naito, Functional Difference Equations with Infinite Delay, Lecture Notes in Math., vol. 1473, Springer-Verlag, Berlin, 1991.

[8] S. Murakami, Representation of solutions of linear functional difference equations in phase space, Nonlinear Anal. 30 (1997) 1153–1164.

[9] S. Murakami, Some spectral properties of the solution operator for linear Volterra difference systems, in: New Developments in Difference Equations and Applications (Taipei, 1997), Gordon and Breach, Amsterdam, 1999, pp. 301–311.

[10] M. Pituk, More on Poincare's and Perron's theorems for difference equations, J. Differ. Equations Appl. 8 (2002) 201–216.